*Journal of Statistical Physics, Vol. 108, Nos. 5/6, September 2002 (© 2002)*

# Hausdorff Dimension in Stochastic Dispersion

**D. Dolgopyat,[1] V. Kaloshin,[2] and L. Koralov[3]**

*Received, November 13, 2001; accepted April 10, 2002*

We consider the evolution of a connected set in Euclidean space carried by a periodic incompressible stochastic flow. While for almost every realization of the random flow at time $t$ most of the particles are at a distance of order $\sqrt{t}$ away from the origin,⁽¹⁾ there is an uncountable set of measure zero of points, which escape to infinity at the linear rate.⁽²⁾ In this paper we prove that this set of linear escape points has full Hausdorff dimension.

**KEY WORDS:** Stochastic flows; Hausdorff dimension; Lyapunov exponents.

## 1. INTRODUCTION

One of the greatest achievements in mathematics of the second half of the last century was creation of the theory of hyperbolic dynamical systems in works of Anosov, Bowen, Ruelle, Sinai, Smale, and many others. The importance of this theory is not so much in that it allows one to get new information about a large class of ordinary differential equations but in that it provides a paradigm for understanding irregular behavior in a large class of natural phenomena. From the mathematical point of view it means that the theory should be useful in many branches of mathematics beyond the study of finite-dimensional dynamical systems. The aim of this note is to illustrate this on a simple example. Namely, we show how the theory of nonuniformly hyperbolic systems, i.e., systems with non-zero Lyapunov

---

Dedicated to our teacher Yakov Sinai on occasion of his 65th birthday.

[1] Department of Mathematics, PennState University, University Park, Pennsylvania 16802; e-mail: dolgop@math.psu.edu

[2] Department of Mathematics, MIT, Cambridge, Massachusetts 02139; e-mail: kaloshin@math.mit.edu

[3] Department of Mathematics, Princeton University, Princeton, New Jersey 08544; e-mail: koralov@math.princeton.edu

0022-4715/02/0900-0943/0 © 2002 Plenum Publishing Corporation

exponents, can explain ballistic behavior in a problem of passive transport in random media.

This paper concerns the long time behavior of a passive substance (say an oil spill) carried by a stochastic flow. Various aspects of such behavior have been a subject of a number of recent papers (see refs. 1–12, etc.) Consider an oil spill at the initial time concentrated in a domain $\Omega$. Let $\Omega$ evolve in time along trajectories of the stochastic flow and $\Omega_t$ be its image at time $t$. The papers mentioned study the rate of stretching of the boundary $\partial\Omega_t$, growth of the diameter and the "shape" of $\Omega_t$, distribution of mass of $\Omega_t$, and many other related questions. In this paper we model the stochastic flow by a stochastic differential equation driven by a finite-dimensional Brownian motion $\{\theta(t)=(\theta_1(t),\dots,\theta_d(t))\in\mathbb{R}^d\}_{t\geqslant 0}$

$$dx_t = X_0(x_t)\,dt + \sum_{k=1}^{d} X_k(x_t)\circ d\theta_k(t) \qquad x\in\mathbb{R}^N \tag{1}$$

where $\{X_k\}_{k=0}^{d}$ are $C^\infty$-smooth space periodic divergence free vector fields on $\mathbb{R}^N$. Alternatively one can regard this system as a flow on $\mathbb{T}^N=\mathbb{R}^N/\mathbb{Z}^N$. Below we impose certain nondegeneracy assumptions on vector fields $\{X_k\}_{k=1}^{d}$ from ref. 1. These assumptions hold on an dense open set of $C^\infty$-smooth divergence free vector fields on $\mathbb{T}^N$ or satisfied generically.

An interesting feature of the flow (1) is the dichotomy between growth of the mass and shape of the spill $\Omega_t$. On one hand, most of the points of the tracer $\Omega_t$ move at distance of order $\sqrt{t}$ at time $t$. More precisely, let $\rho$ be a smooth metric on $\mathbb{T}^N$, naturally lifted to $\mathbb{R}^N$ and $\nu$ be a measure of a finite energy, i.e., for some positive $p$ we have

$$\iint \frac{d\nu(x)\,d\nu(y)}{\rho^p(x,y)} < \infty.$$

In particular, $\nu$ can be the Lebesgue probability measure supported on an open set $\Omega$, which also supports the initial oil spill. Let $\nu_t$ be its image under the flow (1) and $\bar{\nu}_t$ be rescaling of $\nu_t$, defined by as follows: for a Borel set $\Omega\subset\mathbb{R}^N$ put $\bar{\nu}_t(\Omega)=\nu_t(\sqrt{t}\,\Omega)$.

**Theorem 1 (ref. 1).** For almost every realization of the Brownian motion $\{\theta(t)\}_{t\geqslant 0}$ the measure $\bar{\nu}_t$ weakly converges to a Gaussian measure on $\mathbb{R}^N$ as $t\to\infty$.

**Remark 1.** Notice that this is the Central Limit Theorem with respect to randomness in initial conditions, not with respect to randomness of the Brownian motion $\{\theta(t)\}_{t\geqslant 0}$.

On the other hand, there are many points with *linear growth*. Fix a realization of the Brownian motion $\{\theta(t)\}_{t \geqslant 0}$. Let $\mathbf{L}_\theta$ denote the random set of points with a linear escape

$$\mathbf{L}_\theta = \left\{ x \in \mathbb{R}^N : \lim_{t \to +\infty} \inf \frac{|x_t|}{t} > 0 \right\}.$$

The following result is a special case of ref. 10 (see also ref. 2).

**Theorem 2.** Let $S$ be a connected set containing at least two points. Then for almost every realization of the Brownian motion $\{\theta(t)\}_{t \geqslant 0}$ the set $\mathbf{L}_\theta \cap S$ is uncountable.

In fact in dimension 2 there is a limiting shape of the rescaled contaminated area. Namely, let $\Omega$ be a bounded open set, $\Omega_t$ be its image under the flow (1) and $\mathscr{C}_t = \bigcup_{0 \leqslant s \leqslant t} \Omega_s$. In other words, we call a point $x$ *contaminated by the time t* if there is a trajectory from our set $\Omega$ which has passed through $x$ before time $t$.

**The Shape Theorem (ref. 8).** If $N = 2$, then there exists a convex compact set $\mathbb{B} \subset \mathbb{R}^2$ such that for almost every realization of the Brownian motion $\{\theta(t)\}_{t \geqslant 0}$ and any $\delta > 0$ there exists $T = T(\delta)$ such that for all $t > T$

$$(1 - \delta)\, \mathbb{B} \subset \frac{\mathscr{C}_t}{t} \subset (1 + \delta)\, \mathbb{B}.$$

**Remark 2.** The "shape" $\mathbb{B} \subset \mathbb{R}^2$ is independent of the initial spill $\Omega$. Moreover, an open set $\Omega$ can be replaced by a smooth curve $\gamma$ for the Shape Theorem to hold true.

In view of Theorems 1, 2, and the Shape Theorem it is interesting to see how large is the set of points with linear growth. In this paper we first prove the following

**Theorem 3.** Let $\gamma$ be a smooth curve on $\mathbb{R}^2$. Then for almost every realization of the Brownian motion $\{\theta(t)\}_{t \geqslant 0}$ we have $\mathrm{HD}(\mathbf{L}_\theta \cap \gamma) = 1$.

Then in Section 8 using this Theorem we derive the following main result of the paper

**Theorem 4 (Main Result).** For almost every realization of the Brownian motion $\{\theta(t)\}_{t \geqslant 0}$ we have that points of the flow (1) with linear escape to infinity $\mathbf{L}_\theta$ form a dense set of full Hausdorff dimension $\mathrm{HD}(\mathbf{L}_\theta) = N$.

By Theorem 1 for most points $x_0 = x$ in $\mathbb{R}^N$ its trajectory $x_t$ is of order $\sqrt{t}$ away from the origin at time $t$. Also, the Law of Iterated Logarithm for functionals of diffusion processes and Fubini Theorem imply that the set of points $\mathbf{L}_\theta$ with linear escape has measure zero. This Corollary says that $\mathbf{L}_\theta$ is the "richest" possible set of measure zero in $\mathbb{R}^N$, namely, is of full Hausdorff dimension $N$.

## 2. NONDEGENERACY ASSUMPTIONS

In this section we formulate a set of assumptions on the vector fields, which in particular imply the Central Limit Theorem for measures, the estimates on the behavior of the characteristic function of a measure carried by the flow (see ref. 1), and large deviations estimates (see ref. 13). Such estimates are essential for the proof of our results. Recall that $X_0, X_1, \dots, X_d$ are assumed to be $C^\infty$-smooth, periodic and divergence free.

(A) (*hypoellipticity for* $x_t$). For all $x \in \mathbb{R}^N$ we have

$$\mathrm{Lie}(X_1, \dots, X_d)(x) = \mathbb{R}^N.$$

Denote the diagonal in $\mathbb{T}^N \times \mathbb{T}^N$ by

$$\Delta = \{(x^1, x^2) \in \mathbb{R}^N \times \mathbb{R}^N : x^1 = x^2 \ (\mathrm{mod}\ 1)\}.$$

(B) (*hypoellipticity for the two–point motion*). The generator of the two–point motion $\{(x_t^1, x_t^2) : t > 0\}$ is nondegenerate away from the diagonal $\Delta$, meaning that the Lie brackets made out of $(X_1(x^1), X_1(x^2)), \dots, (X_d(x^1), X_d(x^2))$ generate $\mathbb{R}^N \times \mathbb{R}^N$.

To formulate the next assumption we need additional notations. Let $Dx_t : T_{x_0}\mathbb{R}^N \to T_{x_t}\mathbb{R}^N$ be the linearization of $x_t$ at $t$. We need the hypoellipticity of the process $\{(x_t, Dx_t) : t > 0\}$. Denote by $TX_k$ the derivative of the vector field $X_k$ thought as the map on $T\mathbb{R}^2$ and by $S\mathbb{R}^N = \{v \in T\mathbb{R}^N : |v| = 1\}$ the unit tangent bundle on $\mathbb{R}^N$. If we denote by $\tilde{X}_k(v)$ the projection of $TX_k(v)$ onto $T_v S\mathbb{R}^N$, then the stochastic flow (1) on $\mathbb{R}^N$ induces a stochastic flow on the unit tangent bundle $S\mathbb{R}^N$, defined by the following equation:

$$d\tilde{x}_t = \sum_{k=1}^{d} \tilde{X}_k(\tilde{x}_t) \circ d\theta_k(t) + \tilde{X}_0(\tilde{x}_t)\, dt.$$

With these notations we have condition

(C) (*hypoellipticity for* $(x_t, Dx_t)$). For all $v \in S\mathbb{R}^N$ we have

$$\mathrm{Lie}(\tilde{X}_1,\dots,\tilde{X}_d)(v) = T_v S\mathbb{R}^N.$$

For measure-preserving stochastic flows with conditions (C) Lyapunov exponents $\lambda_1,\dots,\lambda_N$ exist by *multiplicative ergodic theorem for stochastic flows* of diffeomorphisms (see ref. 14, Thm. 2.1). Moreover, the sum of Lyapunov exponents $\sum_{j=1}^N \lambda_j$ should be zero (see, e.g., ref. 13). Under conditions (A)–(C) the leading Lyapunov exponent is positive

$$\lambda_1 = \lim_{t \to \infty} \frac{\log |d\varphi_t(x)(v)|}{t} > 0, \tag{2}$$

where $d\varphi_t(x)$ is the linearization matrix of the flow (1) integrated from 0 to $t$ at the point $x$. Indeed, Theorem 6.8 of ref. 15 states that under condition (A) the maximal Lyapunov exponent $\lambda_1$ can be zero only if for almost every realization of the flow (1) one of the following two conditions is satisfied

(a) there is a Riemannian metric $\rho'$ on $\mathbb{T}^N$, invariant with respect to the flow (1) or

(b) there is a direction field $v(x)$ on $\mathbb{T}^N$ invariant with respect to the flow (1).

However (a) contradicts condition (B). Indeed, (a) implies that all the Lie brackets of $\{(X_k(x^1), X_k(x^2))\}_{k=1}^d$ are tangent to the leaves of the foliation

$$\{(x^1, x^2) \in \mathbb{T}^N \times \mathbb{T}^N : \rho'(x^1, x^2) = \mathrm{Const}\}$$

and don't form the whole tangent space. On the other hand (b) contradicts condition (C), since (b) implies that all the Lie brackets are tangent to the graph of $v$. This positivity of $\lambda_1$ is crucial for our approach.

**Remark 3.** Let us mention an important difference between deterministic and stochastic dynamics. Most of the results dealing with statistical properties of deterministic systems assume that all Lyapunov exponents are non-zero. By contrast we need only one positive exponent. This is because in the random situation hypoellipticity condition (C) implies that growth rate of any deterministic vector is given by the largest exponent (see Eq. (19)). This allows us to get our results without assuming that all the exponents are non-zero.

We further require that the flow has no deterministic drift, which is expressed by the following condition

(E) (*zero drift*)

$$\int_{\mathbb{T}^2} \left( \sum_{k=1}^{d} L_{X_k} X_k + X_0 \right)(x)\, dx = 0,$$

where $L_{X_k} X_k(x)$ is the derivative of $X_k$ along $X_k$ at the point $x$. Notice that $\sum_{k=1}^{d} L_{X_k} X_k + X_0$ is the deterministic components of the stochastic flow (1) rewritten in Ito's form.

The Central Limit Theorem for measures was formulated in ref. 1 under an additional assumption

$$\int_{\mathbb{T}^2} X_k(x)\, dx = 0, \qquad k = 1,\dots, d. \tag{3}$$

This assumption is not needed for the proof of Theorem 3 and as the result for the proof of the Main Theorem. However, in order to simplify the proof, i.e., use the results of ref. 1 without technical modifications, we shall assume (3) to hold.

## 3. IDEA OF THE PROOF

### 3.1. A Model Example

Below we define a random dynamical system on $\mathbb{R}$ which models the motion of the projection of the spill $\Omega_t$ onto a fixed line $l \subset \mathbb{R}^N$.

Introduce notations: $I(b; a) = [b - a/2, b + a/2]$—the segment on $\mathbb{R}$ centered at $b$ of length $a$; $s \in \{0, 1\}^{\mathbb{Z}_+}$ a semiinfinite sequence of 0's and 1's, $s_k \in \{0, 1\}^k$ a set of $k$ numbers 0 or 1, $\{\{\theta_{s_k}(t)\}_{s_k \in \{0,1\}^k}\}_{k \in \mathbb{Z}_+}$ countable number of standard i.i.d. Brownian motions on $\mathbb{R}$ indexed by binary sequences. Let $\tau$ be positive.

The random dynamical system is defined as follows. Let $I^{\varnothing} = I(0; 1)$. Then $\sigma_0^\theta \colon I^{\varnothing} \to \mathbb{R}$ stretches $I^{\varnothing}$ uniformly by 2 around its center and shifts it randomly by $\theta_{\varnothing}(\tau)$. Divide $\sigma_0^\theta(I^{\varnothing})$ in two equal parts $I^0$ and $I^1$

$$\sigma_0^\theta(I^{\varnothing}) = I^0 \cup I^1 = I(\theta_{\varnothing}(\tau) - 1/2; 1) \cup I(\theta_{\varnothing}(\tau) + 1/2; 1). \tag{4}$$

Now $\sigma_1^\theta$ acts on each $\{I^i\}_{i=0,1}$ independently by stretching each $I^i$'s uniformly by 2 around its center and shifting by $\theta_0(\tau)$ and $\theta_1(\tau)$ respectively.

$$\begin{aligned}\sigma_1^\theta \circ \sigma_0^\theta(I^\varnothing) &= (I^{00} \cap I^{01}) \cup (I^{10} \cup I^{11})\\ &= I([\theta_\varnothing(\tau)-1/2]+[\theta_0(\tau)-1/2];1)\\ &\quad \cup I([\theta_\varnothing(\tau)-1/2]+[\theta_1(\tau)+1/2];1)\\ &\quad \cup I([\theta_\varnothing(\tau)+1/2]+[\theta_0(\tau)-1/2];1)\\ &\quad \cup I([\theta_\varnothing(\tau)+1/2]+[\theta_1(\tau)+1/2];1),\end{aligned} \tag{5}$$

and so on.

Let $n \in \mathbb{Z}_+$. Then at the $n$-th stage "after time $n\tau$" the image of the initial unit interval $I^\varnothing = [-1/2, 1/2]$ consists of $2^n$ unit intervals. The preimage of each of those unit intervals is an interval of length $2^{-n}$ uniformly contracted. Let's give a different definition of the random dynamical system (4)–(5).

Consider an isomorphism of the dynamical system on the unit interval $I = I^\varnothing + 1/2 = [0, 1]$ given by $\phi\colon x \mapsto 2x \pmod 1$ and the one sided Bernoulli shift on two symbols, say 0 and 1. Such an isomorphism is given by $s\colon x \mapsto s(x) = \{s_k(x)\}_{k=0}^{\infty} \in \{0, 1\}^{\mathbb{Z}_+}$, where for each $k \in \mathbb{Z}_+$

$$\begin{cases} s_k(x) = 0 & \text{if} \quad \phi^n(x) < 1/2 \\ s_k(x) = 1 & \text{otherwise.} \end{cases} \tag{6}$$

Let $\eta_n(x) = \#\{k \leqslant n : s_k(x) = 1\}$. Notice now that

$$\sigma_n^\theta \circ \sigma_{n-1}^\theta \circ \cdots \circ \sigma_0^\theta(x) = \sum_{k=0}^{n} \theta_{s_k}(\tau) + (\eta_{n+1}(x) - (n+1)/2)/2, \tag{7}$$

where $\theta_{s_k}(\tau)$'s are i.i.d. Brownian motions. Define $\eta_-(x) = \liminf_{n\to\infty} \eta_n(x)/n$. Then for almost all points $x \in I$ we have $\eta_-(x) = \lim_{n\to\infty} \eta_n(x)/n = 1/2$. Let us show however that there is full Hausdorff dimension set of points in the interval $I$ such that frequency of 0's is less than frequency of 1's, i.e., $\mathrm{HD}\{x \in I : \eta(x) > 1/2\} = 1$. Since $\sum_{k=0}^{n} \theta_{s_k}(\tau)/n \to 0$ almost surely this would imply that the set of points in $I^\varnothing$ with a nonzero drift for the random dynamical system, defined by (4)–(5), has full Hausdorff dimension almost surely, but is of measure zero.

We shall justify the fact that $\mathrm{HD}\{x \in I : \eta(x) > 1/2\} = 1$.

### 3.2. Points with a Nonzero Drift

Fix an arbitrary small positive $\varepsilon$. The goal is to find a fractal set of points $I_\infty \subset I^\varnothing$ and a probability measure $\mu_\infty$ supported on $I_\infty$ such that $\mu_\infty$-a.e. point $x \in I_\infty$ has a nonzero drift to the right, i.e.,

$\liminf_{n\to\infty} \sigma_n^\theta \circ \ldots \circ \sigma_0^\theta(x)/n > 0$. Moreover, $\mathrm{HD}(\mu_\infty)$ tends to 1 as $\varepsilon$ tends to 0.

Construction of the set $I_\infty$ and of the measure $\mu_\infty$ is inductive. $I_\infty$ is defined as a countable intersection of a nested sequence of compact sets and $\mu_\infty$ is given as a weak limit of Lebesgue measures supported on those sets. We describe the base of the induction and the inductive steps.

- For $n=1$ we have $\sigma_0^\theta(I^\varnothing)$ is a segment of length 2 or union of two segments $I^0$ and $I^1$ of length 1 each. Cut off the bottom $\varepsilon$-segment from each segment. This corresponds to cutting off $\varepsilon$-segments $[-1/2, -1/2+\varepsilon]$ and $[0, \varepsilon]$ from $I^\varnothing$. Denote the surgery result by $I_0 \subset I^\varnothing$ and by $\mu_0$ the Lebesgue probability measure supported on whole $I_0$. Notice that

$$\mu_0\{x \in I_0 : \sigma_1^\theta(x) = 1\} > \mu_0\{x \in I_0 : \sigma_1^\theta(x) = 0\} \tag{8}$$

creates a nonzero drift up, since frequency of 1's exceeds frequency of 0's.

- Suppose $I_{n-1}$ and $\mu_{n-1}$ are constructed. To construct $I_n$ and $\mu_n$ consider the image $\sigma_n^\theta(I_{n-1})$. It consists of $2^n$ segments of equal length close to 1. Cut off the bottom $\varepsilon$-segment from each. This corresponds to cutting off $2^n$ segments of length $2^{-n}\varepsilon$ from $I_{n-1} \subset I^\varnothing$. The result of the surgery is denoted by $I_n$ and by $\mu_n$ we denote the Lebesgue probability measure supported on the whole $I_n$. Again the surgery increases probability of $s_n(x)$ being 1 over $s_n(x)$ being 0. Thus, this creates a positive drift.

The intersection $I_\infty = \bigcap_n I_n$ is a fractal set and the weak limit measure $\mu_\infty = \lim \mu_n$ has Hausdorff dimension approaching 1 as $\varepsilon$ tends to zero. It follows from the construction that for $\mu_\infty$-almost every point $\eta_-(x) = \mu_0\{x \in I_0 : \sigma_1^\theta(x) = 1\} > 1/2$.

## 3.3. Difficulties in Extending of the Model Example to the Case of the Flow

Let $\gamma \subset \mathbb{R}^N$ be a smooth curve, $l \in \mathbb{R}^N$ be a line, and $\pi_l \colon \mathbb{R}^N \to l$ be an orthogonal projection onto $l$. Suppose at the initial moment of time $\pi_l(\gamma) = I^\varnothing$ is the unit interval. If not, then rescale it and shift it to the origin.

The most subtle element in extending the Model Example is defining the stopping (stretching) time $\tau$ or deciding *when to stop* $\gamma_t$ and *how to cut off* some parts of $\gamma_t$ in order to create a nonzero drift as in (8). Such a stopping time needs to have several important features.[4]

---

[4] In the Model Example $\tau$ is a constant.

1. *Stretching property of the stopping time*: It is not difficult to show that if $|\pi_l(\gamma_0)| = 1$, then the stopping time

$$\tau_\gamma = \inf\{t \geqslant 0 : |\pi_l(\gamma_t)| = 2\} \tag{9}$$

has finite expectation and exponential moments uniformly bounded over all compact curves with projection of length 1 (see, e.g., ref. 2).

The analogy between this $\tau_\gamma$ and the model $\tau$ is clear. However, the geometry of $\gamma_{\tau_\gamma}$ in $\mathbb{R}^N$ might become quite complicated ($\gamma_{\tau_\gamma}$ might spin, bend, fold, and so on see computer simulations in ref. 4) so it is not reasonable to stop all parts of the curve $\gamma$ simultaneously and perform the surgery (cut off of "bottom" parts as in the passage preceding (8)). For this reason at the first stage of a partition/cut off process we split $\gamma_{\tau_\gamma}$ not in *two* parts as in the Model Example, but in *a number* (may be countable) of random parts $\gamma = \bigcup_{j \in J} \gamma_j$ and each part $\gamma_j$ will have *its own stopping time* $\tau_j$.

2. *A countable partition of* $\gamma$: We shall partition $\gamma$ into at most countable number of segments $\gamma = \bigcup_{j \in J} \gamma_j$ (see Section 5). Each $\gamma_j$ has its own stopping time $\tau_j$ so that the image $\varphi_{\tau_j} \gamma_j$ under the flow (1) is not too folded (see condition (b) of Theorem 5). Moreover, such a stopping time $\tau_j$ still has finite expectation and exponential moments (see condition (e) of Theorem 5).

Now if we have that the image $\varphi_{\tau_j} \gamma_j$ is "regular" it *does not reflect dynamics* on $\gamma_j$. In order to imitate the Model Example's cut off construction we need to stop $\varphi_t \gamma_j$ at the moment when $\varphi_t|_{\gamma_j}$ is more or less uniformly expanding on $\gamma_j$ forward in time or $(\varphi_t)^{-1}|_{\varphi_t \gamma}$ is uniformly contracting on $\varphi_t \gamma$ backward in time (see conditions (a), (c), and (d) of Theorem 5). For example, if there is no backward contraction by dynamics of $\varphi_{\tau_j}^{-1}$ on $\varphi_{\tau_j} \gamma_j$, then if we cut off an $\varepsilon$-part of $\varphi_{\tau_j} \gamma_j$ its preimage in $\gamma_j$ might not be small compare to length of $\gamma_j$. As we explain in more details below we need this smallness to estimate Hausdorff dimension of remaining points in $\gamma \supset \gamma_j$ after the surgery. The property of uniformness of distortion of a dynamical system is usually called:

3. *A bounded distortion property*: In the Model Example, Section 3.1 we have *uniform* backward contraction of intervals: at stage $n$ (after time $\tau n$) by a factor $2^{-n}$. So, when we cut off an $\varepsilon$-part of an interval at stage $n$, it corresponds to $2^{-n}\varepsilon$-part of the initial segment $I^{\varnothing}$. This remark makes an estimate of Hausdorff dimension of the set $I_\infty$ or of the measure $\mu_\infty$ supported on $I_\infty$ trivial, because the sets $\{I_n\}_{n \in \mathbb{Z}_+}$ have selfsimilar structure. Certainly, this is no longer true for the evolution of $\gamma$ under the flow (1). Some parts of $\gamma$ expanded by $\varphi_t|_\gamma$ expanded more than others. Condition (c) of Theorem 5 makes sure that there is a backward contraction in time and condition (d) of the same theorem says that rate of backward contraction

Holder regularly depends on a point on a short interval $\gamma_j$. Thus, backward contraction is sufficiently uniform on $\gamma_j$'s.

In the theory of deterministic dynamical systems with non-zero Lyapunov exponents the set of points satisfying uniform estimates for forward and backward expansion (as well as uniform estimates for angles between stable and unstable manifolds) are called *Pesin sets* and times when an orbit visits given Pesin set are called *hyperbolic times*. The existence of Pesin sets follows from abstract ergodic theory (see ref. 16). Understanding the geometry of these sets in concrete examples is an important but often difficult task. In this paper we describe some properties of Pesin sets for stochastic flows. This description plays a key role in the proof of Theorem 3 and we also think it can be useful in many other questions about stochastic flows.

In particular let us mention that the estimates similar to ones given in Section 4 play important role in many other questions in the theory of deterministic systems such as periodic orbit estimates(17) and constructions of maximal measures,(18) etc.

Our arguments in this paper are quite similar to refs. 19 and 20 even though the control of the geometry of images of curves is much more complicated in our case. Some interesting formulas for dimensions of non-typical points can be found in ref. 21. We also refer the readers to the survey of ref. 22 and the book of ref. 23 for more results about dimensions of dynamically defined sets.

The rest of the paper is organized as follows. In Section 4 in Theorem 5 we define a stopping time $\tau$ and prove that it has finite expectation and exponential moments. In Section 4.1 we investigate expansion properties of the flow (1) at the stopping time $\tau$ and complete the proof of Theorem 5. Recall that Section 3.2 above was devoted to the construction of points with a nonzero drift. Namely, we need to construct a Cantor set $I_\infty$ and a measure $\mu_\infty$ supported on $I_\infty$ so that $\mu_\infty$-a.e. point has a nonzero drift. First, in Section 5 we present an algorithm of construction of a random Cantor set $I$ inside the initial curve $\gamma$. Then, in Section 6 we define a probability measure $\mu$ supported on $I$ with almost sure nonzero drift. Hausdorff dimension of such a measure is estimated in Section 7. Main Result (Theorem 4) is derived from Theorem 3 in Section 8. Auxiliary lemmas are in the Appendix at the end of the paper.

## 4. HYPERBOLIC MOMENTS. CONTROL OF THE SMOOTHNESS

Introduce notations. Denote by $\varphi_{t_1,t_2}$ a diffeomorphism of $\mathbb{T}^N$, obtained by solving (1) on the time interval $[t_1, t_2]$, and by $\varphi_t$ the diffeomorphism $\varphi_{0,t}$.

The flow (1) can also be thought as the product of independent diffeomorphisms $\{\varphi_{n,n+1}\colon \mathbb{T}^N \to \mathbb{T}^N\}_{n\in\mathbb{Z}_+}$.

Given positive numbers $K$ and $\alpha$ we say that a curve $\gamma$ is $(K,\alpha)$-*smooth* if in the arclength parameterization the following inequality holds

$$\left|\frac{d\gamma}{ds}(s_1)-\frac{d\gamma}{ds}(s_2)\right| \leqslant K\rho^\alpha(s_2,s_1) \qquad \text{for each pair of points} \quad s_1,s_2\in\gamma.$$

In all the inequalities which appear below the distance $\rho$ between the points on $\gamma$ or its images $\varphi_t\gamma$'s is measured in the arclength metric induced on $\gamma$ or $\varphi_t\gamma$ from the ambient space. In order to do not overload notations we omit dependence on $\gamma$ or $\varphi_t\gamma$ when it is clear from the context which curve we use.

The goal of this section is to show that for a sufficiently small $\alpha$ and a sufficiently large $K$, starting with an arbitrary point $x$ on a $(K,\alpha)$-smooth curve $\gamma$, the part of image of this curve in a small neighborhood of the image of $x$ is often smooth. More precisely, we prove the following statement. Let $\lambda_1$ be the largest Lyapunov exponent of the flow (1) which is positive see (2).

**Theorem 5.** For any $0<\lambda_1'<\lambda_1$ there exist sufficiently small $r>0$, $\alpha\in(0,1)$, and sufficiently large $K>0$ and $n_0\in\mathbb{Z}_+$ with the following properties:

For any $(K,\alpha)$-smooth $\gamma$ of length between $\frac{r}{100}$ and $100r$ and each point $x\in\gamma$ there is a stopping time $\tau=\tau(x)$, divisible by $n_0$, such that

(a) $\|d\varphi_\tau|T\gamma(x)\|>100$ and length of the corresponding curve $l(\varphi_\tau\gamma)\geqslant r$;

Denote by $\bar\gamma_r=\bar\gamma_r(x)$ a curve inside $\varphi_\tau\gamma$ of radius $r$ with respect to induced in $\varphi_\tau\gamma$ length centered at $\varphi_\tau(x)$. Then

(b) $\bar\gamma_r$ is $(K,\alpha)$-smooth

and for each pair of points $y_1,y_2\in\bar\gamma_r$ the following holds

(c) for each integer $0\leqslant k\leqslant\frac{\tau}{n_0}$ we have

$$\rho(\varphi_{\tau,\tau-kn_0}\,y_1,\varphi_{\tau,\tau-kn_0}\,y_2)\leqslant e^{-\lambda_1' kn_0}\rho(y_1,y_2);$$

(d) $|\ln\|d\varphi_\tau^{-1}\,|\,T\bar\gamma_r\|\,(y_1)-\ln\|d\varphi_\tau^{-1}\,|\,T\bar\gamma_r\|\,(y_2)|\leqslant \text{Const}\,\rho^\alpha(y_1,y_2)$;

Moreover, for such a stopping time $\tau(x)$ we have

(e) $\mathbb{E}\,\tau(x)\leqslant C_0$; $\mathbb{P}\{\tau(x)>T\}\leqslant C_1e^{-C_2T}$ for any $T>0$, All the above constants depend only on vector fields $\{X_k\}_{k=0}^{d}$ and $\lambda_1$, but independent of the curve $\gamma$.

**Remark 4.** Choosing integer $n_0$ is only for our convenience. Requirement that $\tau$ is divisible by $n_0$ will be used for construction of partition of $\gamma$ in Section 5. This is also indication of flexibility in choice of both constants. The choice of constants 100, 1000, etc. in this paper is more or less arbitrary. Any constant greater than 1 would suffice.

***Proof.*** The leading idea of the proof is that with probability close to 1 for a sufficiently large $n_0$ the diffeomorphism $\varphi_{t,\,t+n_0}\colon \mathbb{T}^N \to \mathbb{T}^N$ gets close to its asymptotic behavior. In particular, the norm of the linearization $\|d\varphi_{t,\,t+n_0}(x)\|$ as the matrix is $\sim \exp(\lambda_1 n_0 + o(n_0))$ as the top Lyapunov exponent predicts. Moreover, the linearization dominates higher order terms of $\varphi_{t,\,t+n_0}(x)$ and, therefore, determines local dynamics in a neighborhood of $x$. Thus, to some extend for large periods of time the flow (1) behaves similarly to uniformly hyperbolic system, for which properties of the Theorem are easy to verify. Now we start the proof.

First we construct a stopping time $\tau$ as a first moments satisfying a certain number of regularity inequalities (see (11)–(15)). This inequalities would include $K$, $r$, $n_0$ and some other parameters. Then we show that for any $\varepsilon > 0$ these parameters can be adjusted so that probability that the number of times each inequality is violated up to time $T$ at least $\varepsilon T$ times decay exponentially in $T$. This would guarantee condition (e). Finally, we show that these inequalities imply conditions (a)–(d) and as the result prove the Theorem. In the Appendix we obtain large deviation estimates necessary for the proof below.

Our first goal is to control distortion of the unit tangent vector to images $\varphi_t\gamma$ of $\gamma$ as time $t$ evolves. Consider a collection of subsets of $\gamma$ indexed by $j$

$$\mathscr{B}_{T,\,jn_0}(x) = \{y \in \gamma : \rho(\varphi_{n_0 k}\, y, \varphi_{n_0 k} x) \leqslant r e^{-\lambda_1{}'(T - n_0 k)} \text{ for } 0 \leqslant k \leqslant j\},$$

where $j$ varies from 1 to $T/n_0$. We would like to find an integer moment of time $\tau$, divisible by $n_0$, such that

$$\varphi_{jn_0}\mathscr{B}_{\tau,\,\tau}(x) \text{ is } (Ke^{\epsilon(\tau - jn_0)}, \alpha)\text{-smooth for all } j = 0, \ldots, \frac{\tau}{n_0}. \tag{10}$$

The rest of this section is devoted to showing that the set of those $T$, divisible by $n_0$ for which (10) holds with $T = \tau$ has density close to 1 if $K$ is sufficiently large. Given $T$ denote by $K_j$ the $\alpha$-Holder norm of $\varphi_{jn_0}\mathscr{B}_{T,\,jn_0}(x)$.

We would like to derive an inductive in $j$ formula relating $K_j$ and $K_{j+1}$ so that in $T/n_0$ steps we get a required statement. Let $z_1, z_2$ be two points on $\varphi_{jn_0} \mathscr{B}_{T,\, jn_0}(x)$ and $r$ is sufficiently small, then

$$\rho(\varphi_{jn_0,\,(j+1)\,n_0} z_1,\, \varphi_{jn_0,\,(j+1)\,n_0} z_2) \geqslant \tfrac{1}{2} \inf_{\varphi_{jn_0} \mathscr{B}_{T,\, jn_0}(x)} \|d\varphi_{jn_0,\,(j+1)\,n_0} \mid T\gamma\|\, \rho(z_1, z_2).$$

Let $d^2\varphi_{jn_0,\,(j+1)\,n_0}$ be the Hessian matrix consisting of second derivatives of the diffeomorphism $\varphi_{jn_0,\,(j+1)\,n_0}$. Assuming now that for each integer $j < \frac{T}{n_0}$ and some $R > 0$ we have

$$\|d^2\varphi_{jn_0,\,(j+1)\,n_0}\| \leqslant R e^{\epsilon(T - jn_0)} \tag{11}$$

and that condition (10) holds true up for each $j \leqslant j^*$. Then we get

$$\begin{aligned} &\rho(\varphi_{jn_0,\,(j+1)\,n_0} z_1,\, \varphi_{jn_0,\,(j+1)\,n_0} z_2) \\ &\qquad \geqslant \tfrac{1}{2} (\|d\varphi_{jn_0,\,(j+1)\,n_0} \mid T\gamma\|\, (\varphi_{jn_0} x) - rRK e^{-(\lambda_1' n_0 - 2\epsilon)(T - j^*)})\, \rho(z_1, z_2). \end{aligned} \tag{12}$$

We would like to prove that (10) holds true for $j = j^* + 1$. Assume also that for each $j < T/n_0$ we have

$$rRK e^{-(\lambda_1' n_0 - 2\epsilon)(T - jn_0)} \leqslant \frac{\|d\varphi_{jn_0,\,(j+1)\,n_0} \mid T\gamma\|\, (\varphi_{jn_0} x)}{4} \tag{13}$$

then we get

$$\rho(\varphi_{j^*n_0,\,(j^*+1)\,n_0} z_1,\, \varphi_{j^*n_0,\,(j^*+1)\,n_0} z_2) \geqslant \frac{\|d\varphi_{j^*n_0,\,(j^*+1)\,n_0} \mid T\gamma\|\, (\varphi_{j^*n_0} x)}{4}\, \rho(z_1, z_2).$$

Let $v_1$ and $v_2$ be directions of the tangent vectors to $\varphi_{j^*n_0}\gamma$ at $z_1$ and $z_2$ respectively, then

$$\begin{aligned} &\rho(d\varphi_{j^*n_0,\,(j^*+1)\,n_0}(z_1)\, v_1,\, d\varphi_{j^*n_0,\,(j^*+1)\,n_0}(z_2)\, v_2) \\ &\qquad \leqslant \rho(d\varphi_{j^*n_0,\,(j^*+1)\,n_0}(z_1)\, v_1,\, d\varphi_{j^*n_0,\,(j^*+1)\,n_0}(z_1)\, v_2) \\ &\qquad\quad + \rho(d\varphi_{j^*n_0,\,(j^*+1)\,n_0}(z_1)\, v_2,\, d\varphi_{j^*n_0,\,(j^*+1)\,n_0}(z_2)\, v_2). \end{aligned}$$

Denote the first and the second terms by $I$ and $II$ respectively. If (11) holds, then

$$II \leqslant R e^{\epsilon(T - j^* n_0)} \rho(z_1, z_2).$$

Now since $v_1$ and $v_2$ are close $T(\varphi_{j^*n_0}\gamma)(x)$ and $z_1$ is close to $\varphi_{j^*n_0} x$ we have

$$[d\varphi_{j^*n_0,\,(j^*+1)\,n_0}(z_1)\, v_1 - d\varphi_{j^*n_0,\,(j^*+1)\,n_0}(z_1)\, v_2] \approx d_v d\varphi_{j^*n_0,\,(j^*+1)\,n_0}(x)(v_1 - v_2).$$

Let us give more precise estimates. Notice that if $A$ is a linear map, then its action on the projective space satisfies

$$\|dA(v)\,\delta v\| = \frac{\|\Pi_{(Av)^\perp} A\delta v\|}{\|Av\|} \leqslant \frac{\|A\|}{\|Av\|},$$

where $\Pi_{(Av)^\perp}$ is the orthogonal projection onto the direction $Av$ and $\delta v$ is an element of $T_v T_x M$.

Therefore, we can assume that for a positive integer $T/n_0$ and each $j^* < T/n_0$ the following inequality is satisfied

$$\frac{\rho(Av_1, Av_2)}{\rho(v_1, v_2)} \leqslant 2\,\frac{\|d\varphi_{j^*n_0,\,(j^*+1)\,n_0}\|}{\|d\varphi_{j^*n_0,\,(j^*+1)\,n_0} \mid T\varphi_{j^*n_0}\gamma\|}, \tag{14}$$

for any linear map $A$ such that $\|A - d\varphi_{j^*n_0,\,(j^*+1)\,n_0}\| \leqslant rRe^{(\lambda_1{}'n_0-\epsilon)(T-j^*n_0)}$ and for any pair $(v_1, v_2)$ of tangent vectors such that $\rho(v_1, v_2) \leqslant Ke^{-(\lambda_1{}'n_0\alpha-\epsilon)(T-j^*n_0)}$.

Thus

$$I \leqslant 2K_{j^*}\rho^\alpha(z_1, z_2)\,\frac{\|d\varphi_{j^*n_0,\,(j^*+1)\,n_0}\|}{\|d\varphi_{j^*n_0,\,(j^*+1)\,n_0} \mid T\varphi_{j^*n_0}\gamma\|}.$$

Hence (11)–(14) imply that

$$K_{j^*+1} \leqslant \frac{8K_{j^*}\,\|d\varphi_{j^*n_0,\,(j^*+1)\,n_0}\|}{\|d\varphi_{j^*n_0,\,(j^*+1)\,n_0} \mid T\varphi_{j^*n_0}\gamma\|^{1+\alpha}} + \frac{2Re^{-(\lambda_1{}'n_0(1-\alpha)-\epsilon)(T-j^*n_0)}}{\|d\varphi_{j^*n_0,\,(j^*+1)\,n_0} \mid T\varphi_{j^*n_0}\gamma\|^{\alpha}}$$

If $T$ is chosen so that

$$\|d\varphi_{j^*n_0,\,(j^*+1)\,n_0} \mid T\varphi_{j^*n_0}\gamma\| \geqslant (Re^{\epsilon(T-j^*n_0)})^{-1}, \tag{15}$$

then the last inequality becomes

$$K_{j^*+1} \leqslant \frac{8K_{j^*}\,\|d\varphi_{j^*n_0,\,(j^*+1)\,n_0}\|}{\|d\varphi_{j^*n_0,\,(j^*+1)\,n_0} \mid T\varphi_{j^*n_0}\gamma\|^{1+\alpha}} + 2R^2e^{-(\lambda_1{}'n_0(1-\alpha)-2\epsilon)(T-j^*n_0)} \tag{16}$$

Let us summarize what we have learned so far.

**Lemma 1.** For $n_0$ as above suppose that $T$ is such that for every $j$ such that $jn_0 \leqslant T$ estimates (11)–(15) hold true and also the solution of

$$\bar{K}_{j+1} = \frac{4\bar{K}_j\,\|d\varphi_{jn_0,\,(j+1)\,n_0}\|}{\|d\varphi_{jn_0,\,(j+1)\,n_0} \mid T\varphi_{jn_0}\gamma\|^{1+\alpha}} + 2R^2, \qquad \bar{K}_0 = K \tag{17}$$

satisfies

$$\bar{K}_j \leqslant K e^{(T - jn_0)\epsilon} \tag{18}$$

then inequality (10) holds.

Now we want to show that the set of points where either (11)–(15) or (18) fail has density less than $\varepsilon T$ except on a set of exponentially small probability. The result for (18) follows from Proposition 8 from Appendix A applied to $\ln \bar{K}_j$. To see that the conditions of this proposition are satisfied if $\alpha$ is sufficiently small it is enough to verify that $\ln \bar{K}_j$ has uniform drift to the left.

By Carverhill's extension of Oseledets' Theorem$^{(14)}$ for every point $x$ on $M$ and every unit vector $v$ in $T_x M$

$$\frac{1}{n_0} \mathbb{E} \ln \|d\varphi_{n_0}(x)\, v\| \to \lambda_1 \tag{19}$$

uniformly as $n_0 \to \infty$ and ref. 13 provides exponential estimate for probabilities of large deviations. Since

$$\|d\varphi_{n_0}(x)\| \leqslant \sum_{j=1}^{N} \|d\varphi_{n_0}(x)\, v_j\|$$

where $\{v_j\}_{j=1}^{N}$ is any orthonormal frame, the above mentioned results of ref. 13 imply that

$$\frac{1}{n_0} \mathbb{E} \ln \|d\varphi_{n_0}(x)\| \to \lambda_1 \qquad \text{as} \quad n_0 \to \infty$$

with exponential bound for large deviations. Thus Proposition 8 from the Appendix applies to $\ln \bar{K}_j$ for a large enough $n_0$.

The fact that (11)–(15) fail rarely if $n_0$ is sufficiently large and $r$ is sufficiently small follows from Lemma 10.

### 4.1. Hyperbolic Moments. Control of Expansion

We now define the stopping time $\tau$ as the first moment when (10), (11), and (15) are satisfied as well as

$$\|d\varphi_\tau \mid T\gamma\|\,(x) \geqslant 1000 \tag{20}$$

and for each positive integer $j \leqslant \tau / n_0$ and some constant $0 < \tilde{\lambda}_1 < \lambda_1$ we have

$$\|d\varphi_{\tau,\, \tau - jn_0} \mid T\varphi_\tau \gamma\| \leqslant e^{-\tilde{\lambda}_1 jn_0}. \tag{21}$$

Then the large deviations estimates of ref. 13 guarantee that property (20) has density close to 1.

**Lemma 2.** For any $\varepsilon > 0$ and any $0 < \tilde{\lambda}_1 < \lambda_1$ there exists a positive integer $n_0$ such that with probability exponentially approaching to 1 the fraction of integers $\tau$, divisible by $n_0$, with the linearization $d\varphi_{\tau,\tau-jn_0}\,|\,T\varphi_\tau\gamma$ contracting exponentially backward in time for all integer $j$ between 0 and $\tau/n_0$ tends to 1. More precisely,

$$\mathbb{P}\left\{\frac{\#\{S \leqslant L : \forall 0 \leqslant j \leqslant S, \tau = Sn_0\ \|d\varphi_{\tau,\tau-jn_0}\,|\,T\varphi_\tau\gamma\| \leqslant e^{-\tilde{\lambda}_1 jn_0}\}}{L} \leqslant 1-\varepsilon\right\}$$

decays exponentially in $L$.

*Proof.* We first show how to prove a weaker statement with "$\exists\varepsilon$" instead of "$\forall\varepsilon$" (which is enough to prove Theorem 5) and then explain briefly the changes needed to prove the sharp result.

Let $\tau_1$ be the first moment such that for each integer $j \leqslant \frac{\tau}{n_0}$

$$\|d\varphi_{\tau_1,\tau_1-jn_0}\,|\,T\varphi_\tau\gamma\| \leqslant e^{-\tilde{\lambda}_1 jn_0}. \tag{22}$$

We claim that $\tau_1$ has exponential tail. Indeed, let

$$Y_j = Y_j(\theta) = (\|d\varphi_{jn_0}\,|\,T\gamma\|\,(x)\,e^{-(\tilde{\lambda}_1+\varepsilon)\,jn_0})^\theta, \qquad Y_0 = 1, \qquad \text{and}$$

$$Z_j = \|d\varphi_{jn_0}\,|\,T\gamma\|\,(x)\,e^{-\tilde{\lambda}_1 jn_0}, \qquad Z_0 = 1.$$

Then ref. 13 shows that if $n_0$ is sufficiently large and $\varepsilon, \theta$ are sufficiently small, then $Y_j$ is a submartingale. Thus the first moment $\hat{j}$ such that $Z_{\hat{j}} > 10$ has exponential tail. But there is at least one maximum $\bar{j}$ of $Z_j$ between 0 and $\hat{j}$. Then $\bar{j}$ satisfies (22).

Now define $\tau_k$ inductively so that $\tau_{k+1} > \tau_k$ is the first moment such that for every $j \leqslant \frac{\tau_{k+1}-\tau_k}{n_0}$

$$\|d\varphi_{\tau_{k+1},\tau_{k+1}-jn_0}\,|\,T\varphi_{\tau_{k+1}}\gamma\| \leqslant e^{-\tilde{\lambda}_1 n_0 j}.$$

Then $\tau_{k+1}-\tau_k$ have exponential tails, so by Lemma 9 there exists $c$ such that $\mathbb{P}\{\frac{\tau_k}{k} \geqslant C\}$ decays exponentially in $k$. However all $\tau_k$ satisfy (22). This proves the result with $\varepsilon = 1-\frac{1}{C}$. To get the optimal result one should note that $\mathbb{P}\{\tau_1 = n_0\} \to 1$ as $n_0 \to \infty$ and apply the arguments of Lemma 9. We leave the details to the reader. ▮

Now we want to verify conditions (b), (c), and (d) of Theorem 5 with $\bar{\gamma}_r$ replaced by $\hat{\gamma} = \varphi_\tau B_{\tau,\tau}(x)$. Once we prove this we get from (c) that the main restriction on $B_{\tau,\tau}(x)$ is for $k=\tau$ so that $\bar{\gamma}_r=\hat{\gamma}$ and then (a) will also be true. Now (b) is true by Lemma 1. We will establish (c) and (d) by induction. Namely we suppose that (c) is true for $k \geqslant k_0$. Then for every $y \in \hat{\gamma}$

$$\begin{aligned}
&|\ln \|d\varphi_{\tau,\tau-(k_0+1)n_0}\,|T\hat{\gamma}\|\,(x)-\ln \|d\varphi_{\tau,\tau-(k_0+1)n_0}\,|T\hat{\gamma}\|\,(y)| \\
&\quad\leqslant \sum_{m=0}^{k_0} |\ln \|d\varphi_{\tau-mn_0,\tau-(m+1)n_0}\,|T\hat{\gamma}\|\,(x)-\ln \|d\varphi_{\tau-mn_0,\tau-(m+1)n_0}\,|T\hat{\gamma}\|\,(y)| \\
&\quad\leqslant \sum_{m=0}^{k_0} |\ln \|d\varphi_{\tau-mn_0,\tau-(m+1)n_0}\,|T\hat{\gamma}\|\,(x)-\ln \|d\varphi_{\tau-mn_0,\tau-(m+1)n_0}\,|T\hat{\gamma}\|\,(y)| \\
&\qquad+\sum_{m=0}^{k_0} |\ln \|d\varphi_{\tau-mn_0,\tau-(m+1)n_0}\,|T\hat{\gamma}\|\,(y)-\ln \|d\varphi_{\tau-mn_0,\tau-(m+1)n_0}\,|T\hat{\gamma}\|\,(y)|.
\end{aligned}$$

Denote the first term by $I$ and the second term by $II$ respectively. Now by (10) and (11)

$$\begin{aligned}
I &\leqslant \sum_{m=0}^{k_0} Re^{\epsilon m}Ke^{\epsilon m}\rho^{\alpha}(\varphi_{\tau,\tau-mn_0}x,\varphi_{\tau,\tau-mn_0}y) \\
&\leqslant \sum_{m=0}^{k_0} KRe^{-(\lambda_1{}'\alpha n_0-2\epsilon)m}\rho^{\alpha}(x,y) \leqslant \text{Const } r^{\alpha}.
\end{aligned} \tag{23}$$

On the other hand

$$\begin{aligned}
II &\leqslant \sum_{m=0}^{k_0} \frac{\|d^2\varphi_{\tau-mn_0,\tau-(m+1)n_0}\|}{\|d\varphi_{\tau-mn_0,\tau-(m+1)n_0}\|}\,\rho(\varphi_{\tau,\tau-mn_0}x,\varphi_{\tau,\tau-mn_0}y) \\
&\leqslant \sum_{m=0}^{k_0} R^2e^{-(\lambda_1'n_0-2\epsilon)m}\rho(x,y) \leqslant \text{Const } r.
\end{aligned} \tag{24}$$

Hence (11) and (15)

$$|\ln \|d\varphi_{\tau,\tau-(k_0+1)n_0}\,|T\hat{\gamma}\|\,(x)-\ln \|d\varphi_{\tau,\tau-(k_0+1)n_0}\,|T\hat{\gamma}\|\,(y)| \leqslant C(R)\,\rho(y_1,y_2). \tag{25}$$

Thus for all $y$

$$\|d\varphi_{\tau-(k_0+1)n_0,\tau}\,|T\varphi_{\tau-(k_0+1)n_0}\gamma\|\,(y) \geqslant \exp(\tilde{\lambda}_1kn_0-C(R)\,r) \geqslant \exp(\lambda_1'kn_0) \tag{26}$$

if $\tilde{\lambda}_1-\lambda_1' \geqslant C(R)\,r$. (26) implies that (c) is valid for $k_0-1$. Thus, we obtain (c) for all $k$. Now repeating the proof of (25) with $x$ and $y$ replaced by $y_1$ and $y_2$ (and using (26) instead of (21)) we obtain (d). This completes the proof of Theorem 5. ▌

**Remark 5.** The term *hyperbolic time* was introduced in ref. 24 but the notion itself was used before, e.g., in refs. 16 and 25–27. Considerations of this section are similar to refs. 28 and 29 but the additional difficulty is that in those papers the analogue of (10) was true by the general theory of partially hyperbolic systems[(30)] whereas here additional arguments in spirit of refs. 16 and 25 were needed to establish it.

One interesting question is how large can $\alpha$ be so that Theorem 5 still holds. We note that $\alpha$ appears in (16) twice. So we want $\alpha$ to be as large as possible to control the first part and we want $\alpha$ to be small to control the second term. In general, the optimal choice of $\alpha$ should depend on the ratio of leading exponents. We refer to refs. 31–34 for the discussion of this question.

## 5. CONSTRUCTION OF THE PARTITION

We are now ready to describe a partition $\gamma = \bigcup_{j\in\mathbb{Z}_+} \gamma(j)$. It will be defined inductively. Each of $\gamma(j)$'s is a finite union of intervals. As $j$ tends to infinity size of intervals tends to zero and they fill up $\gamma$. To simplify the notation we assume that Theorem 5 is true with $n_0 = 1$. This can be achieved by rescaling the time. Fix an orientation from left to right on $\gamma$.

Suppose $\gamma(1), \gamma(2),\ldots, \gamma(m)$ are already defined in an $\mathscr{F}_m$-measurable way. Let

$$K_{m+1} = \{x \in \gamma : \tau(x) = m+1\}. \tag{27}$$

By definition $K_{m+1}$ is a finite union of intervals. Let $U_{m+1} = \varphi_{m+1} K_{m+1}$. We call *an obstacle* any point on the boundary of either $K_{m+1}$, $\bigcup_{j=1}^m \gamma(j)$ or $\gamma$. Fix $r$ satisfying Theorem 5. Let $C$ be a connected component of $U_{m+1}$ and $a$ and $b$ be its left and right endpoints with respect to left-right orientation induced by $\varphi_{m+1}$. If distance from $b$ to the closest image of an obstacle to the right on $\varphi_{m+1}(\gamma)$ is less than $\frac{r}{2}$ and $b'$ is this image, then put $\tilde{b} = b'$. Otherwise let $\tilde{b}$ be a point at distance $\frac{r}{100}$ from $b$. Define $\tilde{a}$ similarly. Consider the set $W_{m+1} = \bigcup_C \tilde{a}\tilde{b}$. Divide $W_{m+1}$ into the segments of lengths between $\frac{r}{100}$ and $\frac{r}{50}$ and denote this partition by $V_{m+1}$. Now we define partition of a subset of $\gamma \setminus \bigcup_{j=1}^m \gamma(j)$ by pulling back along $\varphi_{m+1}^{-1}$ the partition $V_{m+1}$

$$\gamma(m+1) = \varphi_{m+1}^{-1} V_{m+1}. \tag{28}$$

To justify that this algorithm produces a partition which covers all of $K_{m+1}$ we need to check that length of each component is at least $\frac{r}{100}$. To do this we argue by contradiction. Otherwise, there would be two obstacles $x'$, $x''$

neither of which is from $K_{m+1}$ such that $\rho(\varphi_{m+1}x', \varphi_{m+1}x'') \leqslant \frac{r}{100}$ and a point from $U_{m+1}$ between them. At least one of the obstacles would have to come from $\bigcup_{j=1}^{m} \gamma(j)$. Let $x'$ be such an obstacle. Since both points are close to $U_m$ for each $n \leqslant m+1$ we have

$$\rho(\varphi_n x', \varphi_n x'') \leqslant \frac{r}{100}\, e^{-\lambda_1{}'(m-n)}.$$

But in this case the interval $[x', x'']$ in $\gamma$ with endpoints $x'$ and $x''$ would be added to our partition at a previous step of the algorithm.

Denote by $\gamma = \bigcup_{j \in \mathbb{Z}_+} \gamma_j$ the partition which is made out of the partition $\gamma = \bigcup_{j \in \mathbb{Z}_+} \gamma(j)$ by renumerating intervals of this partition in length decreasing order. Let us summarize the outcome.

**Proposition 6.** We can partition $\gamma = \bigcup_{j \in \mathbb{Z}_+} \gamma_j$ in such a way that

(a) there exists a positive integer $n_j$ such that $\|d\varphi_{n_j} \mid T\gamma\| \geqslant 100$ and length $l(\varphi_{n_j}\gamma_j) \geqslant \frac{r}{100}$ (see Remark 4);

(b) for each positive integer $m \leqslant n_j$ and lengths of the corresponding curves we have $l(\varphi_m \gamma_j) \leqslant l(\varphi_{n_j}\gamma_j)\, e^{-\lambda_1{}'(n_j - m)}$;

(c) $|\ln \|d\varphi_{n_j} \mid T\gamma\|\,(x') - \ln \|d\varphi_{n_j} \mid T\gamma\|\,(x'')| \leqslant \text{Const}\, \rho^{\alpha}(\varphi_{n_j}x', \varphi_{n_j}x'')$ for every pair $x', x'' \in \gamma_j$;

(d) for some $\alpha > 0$ and each pair $x', x'' \in \gamma_j$ we have $|v(x', n_j) - v(x'', n_j)| \leqslant \text{Const}\, \rho^{\alpha}(\varphi_{n_j}x', \varphi_{n_j}x'')$ , where $v(x, n)$ denote the unit tangent vector to $\varphi_n \gamma$ at $\varphi_n(x)$;

(e) Let $j(x)$ be such that $x \in \gamma_{j(x)}$. Then $\mathbb{E} n_{j(x)} \leqslant \text{Const}$ and $\mathbb{P}\{n_{j(x)} > T\} \leqslant C_1 e^{-C_2 T}$ for some positive $C_1$, $C_2$ and any $T > 0$;

This Proposition is designed to allow application of Theorem 5 so that we can use regularity and geometric properties of $\gamma_j$'s at stopping times $\tau_j$'s.

## 6. CONSTRUCTION OF THE MEASURE WITH ALMOST SURE NONZERO DRIFT

Now we construct a random Cantor set $I \subset \gamma$ and a probability measure $\mu$ supported on $I$ such that $\mu$—almost all points have a nonzero drift. This construction goes along the same line with the construction in Section 3.2 of the Cantor set $I_\infty$ in the unit interval and a probability measure $\mu_\infty$ on $I$ such that $\mu_\infty$—almost all points have nonzero drift.

Choose a direction $\vec{e} \in \mathbb{R}^N$. Let $\theta$ be a small parameter which we let to zero in the next section. We say that a curve is *$\vec{e}$-monotone* if its projection to $\vec{e}$ is monotone. Now we describe construction of a Cantor set $I \subset \gamma$ and a probability measure $\mu$ on $I$ by induction. This Cantor set $I$ at $k$-th step of induction consists of countable number of segments numerated by $k$-tuples of positive integers.

Denote $k$-tuples $(j_1,\dots, j_k) \in \mathbb{Z}_+^k$ and $(n_1,\dots, n_k) \in \mathbb{Z}_+^k$ by $J_k$ and $N_k$ respectively. Let $|N_k| = \sum_{j=1}^k n_j$.

The first step of induction goes as follows. Let $\gamma_j$, $n_j$ be the sequence of pairs: a curve and an integer, described in Proposition 6. Let $\theta$ be a small positive number. If $\varphi_{n_j}\gamma_j$ is $\vec{e}$-monotone put $\sigma(j)$ equal $\varphi_{n_j}\gamma_j$ without the segment of length $\theta r$, which we cut off from the $\vec{e}$-bottom point of $\varphi_{n_j}\gamma_j$. Otherwise $\sigma(j) = \varphi_{n_j}\gamma_j$ with no cut off. Let $\gamma(J_1) = \varphi_{n_j}^{-1}\sigma(j)$ and $N_1(J_1) = n_j$ for $J_1 = j$.

Suppose a collection of disjoint segments $\{\gamma(J_k)\}_{J_k \in \mathbb{Z}_+^k} \subset \gamma$ is defined as above and multiindices $N_k$ (resp. $J_k$) are defined as the corresponding set of hyperbolic times multiindexed by $J_k$ segments. Then

$$I_k = \bigcup_{J_k \in \mathbb{Z}_+^k} \gamma(J_k) \subset I_{k-1} \subset \cdots \subset I_1 \subset \gamma \tag{29}$$

is the $k$-th order of construction of the random Cantor set $I$ (cf. with an open set $I_k$ from Section 3.2).

The $(k+1)$-st step goes as follows. Pick a segment $\gamma(J_k)$ of partition (29). Consider the partition of the curve

$$\varphi_{|N_{J_k}|}\gamma(J_k) = \bigcup_{j_{k+1} \in \mathbb{Z}_+} \tilde{\gamma}(J_k, j_{k+1}) \tag{30}$$

defined in Section 5 and let $n_{J_k, j_{k+1}}$ be the corresponding hyperbolic times for $\tilde{\gamma}(J_k, j_{k+1})$ from Proposition 6. For brevity denote $|N_k(J_k)|$ by $n^{(k)}$ and $|N_k(J_k)| + n_{(J_k, j_{k+1})}$ by $n^{(k+1)}$. If the curve $\varphi_{n^{(k)}, n^{(k+1)}}\tilde{\gamma}_{(J_k, j_{k+1})}$ is $\vec{e}$-monotone we let $\sigma(J_k, j_{k+1})$ be $\varphi_{n^{(k)}, n^{(k+1)}}\tilde{\gamma}_{(J_k, j_{k+1})}$ with cut off of the segment of length $\theta$ starting from the $\vec{e}$-bottom. Otherwise, $\sigma(J_k, j_{k+1})$ equal $\varphi_{n^{(k)}, n^{(k+1)}}\tilde{\gamma}(J_k, j_{k+1})$ with no cut off. Then a segment

$$\gamma(J_k, j_{k+1}) = \varphi_{n^{(k+1)}}^{-1}\sigma(J_k, j_{k+1}) \tag{31}$$

with $j_{k+1} \in Z_+$ this defines the $(k+1)$-st order partition $\{\gamma(J_{k+1})\}_{J_{k+1} \in \mathbb{Z}_+^{k+1}} \subset \gamma$ and the $k$-order set $I_{k+1} = \bigcup_{J_{k+1} \in \mathbb{Z}_+^{k+1}} \gamma(J_{k+1}) \subset \gamma$.

We now describe a sequence of measures $\mu_k$'s on $I_k \subset \gamma$ with $k \in \mathbb{Z}_+$ respectively. Let $\mu_0$ be the arclength on $\gamma$. Suppose $\mu_k$ is already defined on $I_k$. Consider $\{\gamma(J_{k+1})\}_{J_{k+1} \in \mathbb{Z}_+^{k+1}}$. If $\varphi_{n^{(k+1)}}\gamma(J_{k+1})$ is *not* $\vec{e}$-monotone we let $\mu_{k+1}|_{\gamma(J_{k+1})} = \mu_k|_{\gamma(J_{k+1})}$. Otherwise, $\mu_{k+1}|_{\gamma(J_{k+1})} = \rho_{jk}\mu_k|_{\gamma(J_{k+1})}$, where $\rho_{jk}$ is a normalizing constant.

**Lemma 3.** Let $k$ be an integer. If $r$ is sufficiently small and $\gamma \subset \mathbb{R}^N$ is $(K, \alpha)$-smooth as in Theorem 5, then by partition of $\gamma$ up to order $k+1$, each multiindex $J_k \in \mathbb{Z}_+^k$ the corresponding $k$-th order curve $\gamma(J_k) \subset \gamma$ satisfy the property: for any positive integer $j_{k+1}$ the $(k+1)$-st order curve $\gamma(J_{k+1}) \subset \gamma(J_k)$ has $\vec{e}$-monotone with positive probability, i.e.,

$$\mathbb{P}\{\varphi_{n^{(k+1)}}\gamma(J_{k+1}) \text{ is } \vec{e}\text{-monotone} \,|\, \mathscr{F}_{n^{(k)},\, n^{(k+1)}}\} > c$$

for some positive $c$ and $c$ is uniform for all $(K, \alpha)$-smooth curves.

*Proof.* Pick a point $x \in \gamma(J_{k+1})$. By assumption (D) of hypoellipticity on the unit tangent bundle $SM$ for the flow (1) probability that the angle between $\vec{e}$ and $T\varphi_{n^{k+1}}\gamma(x)$ makes less than 1° is positive. By definition $\varphi_{n^{(k+1)}}\gamma(J_{k+1})$ is $(K, \alpha)$-smooth. Thus if $r$ is small enough, then the tangent vectors to $\varphi_{n^{k+1}}\gamma$ are close to $T\varphi_{n^{k+1}}\gamma(x)$ with large probability, where $x$ is a point on $\gamma(J_{k+1})$. This completes the proof. ▮

Recall that $\theta > 0$ is a fraction of $\varphi_{n_j}\gamma_j$ we cut off from $\varphi_{n_j}\gamma_j$ on the $j$-th step, provided $\varphi_{n_j}\gamma_j$ is $\vec{e}$-monotone. Let $\mu = \mu(\theta)$ denote the weak limit of $\mu_k$'s

$$\mu = \lim_{k \to \infty} \mu_k.$$

**Lemma 4.** For almost every realization of the Brownian motion $\{\theta(t)\}_{t \geqslant 0}$ and $\mu$—almost every $x$

$$\liminf_{t \to \infty} \frac{\langle x_t, e\rangle}{t} > 0.$$

*Proof.* The first step is to show that for any $s$ for almost all realizations of the Brownian motion $\{\theta(t)\}_{t \geqslant 0}$

$$\liminf_{t \to \infty} \frac{\langle x^{(k)}_{n^{(k)}}, e\rangle}{n^{(k)}} > 0 \tag{32}$$

Applying Proposition 6(e) and Lemma 9 we get that there exists a constant $C > 0$ such that

$$\limsup_{k \to \infty} \frac{n^{(k)}}{k} < C$$

almost surely. Therefore, to prove (32) it suffices to show that

$$\liminf_{k \to \infty} \frac{\langle x^{(k)}_{n^{(k)}}, e\rangle}{k} > 0 \tag{33}$$

However by Lemma 3 there exists $c$ such that $\mathbb{E}\langle x^{(k+1)}_{n^{(k+1)}} - x^{(k)}_{n^{(k)}}, e\rangle > c$ uniformly in $k, s$. (This is because $\mathbb{E}\langle x^{(k)}_{n^{(k+1)}} - x^{(k)}_{n^{(k)}}, e\rangle = 0$ and

$$\langle x^{(k)}_{n^{(k+1)}}, e\rangle - \langle x^{(k)}_{n^{(k)}}, e\rangle \geqslant 0$$

with strict inequality having positive probability by Lemma 3.) Hence (33) follows by Lemma 9. Therefore (32) is established. ▮

Now we apply the following estimate.

**Lemma 5 (ref. 7, Thm. 1).** Let

$$\Phi_{s,t} = \sup_{s \leqslant \tau \leqslant t} |x_\tau - x_s|, \qquad \tilde{\Phi}_{s,t} = \frac{\Phi_{s,t}}{\max(1, t-s)}$$

then there exists a constant $C$ such that for all $s$ and $t$

$$\mathbb{E}\left(\exp\left\{\frac{\tilde{\Phi}^2_{s,t}}{\max(1, \ln^3 \tilde{\Phi}_{s,t})}\right\}\right) < C.$$

Combining this lemma with Proposition 6(e) we obtain that there are positive constants $\alpha$ and $D$ such that

$$\mathbb{E}(\exp\{\alpha \sup_{n^{(k)} < \tau < n^{(k+1)}} |x_\tau(s) - x_{n^{(k)}}|\}) < D.$$

Using Borel–Cantelli's lemma we derive from this that almost surely

$$\limsup_{k \to \infty} \frac{\sup_{n^{(k)} < \tau < n^{(k+1)}} |x_\tau(s) - x_{n^{(k)}}|}{\ln k} < +\infty.$$

Therefore for any $s$ and for almost all realizations of the Brownian motion $\{\theta(t)\}_{t \geqslant 0}$ we have

$$\liminf_{\tau \to \infty} \frac{\langle x_\tau(s), e\rangle}{\tau} > 0.$$

By Fubini Theorem we have that for almost every realization of the Brownian motion $\{\theta(t)\}_{t \geqslant 0}$ the set

$$\left\{s\colon \liminf_{\tau \to \infty} \frac{\langle x_\tau(s), e\rangle}{\tau} > 0\right\}$$

has full measure. ▮

## 7. HAUSDORFF DIMENSION OF $\mu$

In this section we complete the proof of Theorem 3 by establishing the following fact. Recall that $\theta > 0$ is a fraction of $\varphi_{n_j}\gamma_j$ we cut off from $\varphi_{n_j}\gamma_j$ on the $j$-th step, provided $\varphi_{n_j}\gamma_j$ is $\vec{e}$-monotone. Consider the measure $\mu$ we constructed in the previous Section.

**Proposition 7.** With notations above we have that as $\theta \to 0$ Hausdorff dimension of the measure $\mu = \mu(\theta)$ tends to 1: $\mathrm{HD}(\mu(\theta)) \to 1$.

Let us recall the following standard principle.

**Lemma 6 (Mass Distribution Principle).** Let $S$ be a compact subset of a Euclidean (or metric) space such that there exists a probability measure $\nu$ such that $\nu(S) = 1$ and for each $x$ we have $\nu(\mathscr{B}(x, r)) \leqslant Cr^s$ for some positive $C$ and $s$. Then $\mathrm{HD}(S) \geqslant s$.

Proposition 7 is a direct consequence of the following statements.

**Lemma 7.** Let $\gamma$ be a smooth curve in $\mathbb{R}^N$. Suppose there exist a nested sequence of partitions

$$\gamma \supset \bigcup_{J_1 \in \mathbb{Z}_+^1} \gamma(J_1) \supset \cdots \supset \bigcup_{J_k \in \mathbb{Z}_+^k} \gamma(J_k) \supset \cdots \tag{34}$$

and probability measures $\mu_0, \mu_1,..., \mu_k,...$ supported on $\gamma$, $\bigcup_{J_1 \in \mathbb{Z}_+^1} \gamma(J_1),...,$ $\bigcup_{J_k \in \mathbb{Z}_+^k} \gamma(J_k),...$ respectively such that $\mu_0$ is the normalized arclength on $\gamma$ and so on $\mu_k$ is the normalized arclength on $\bigcup_{J_k \in \mathbb{Z}_+^k} \gamma(J_k)$. Suppose we have

(a) for all $J_k \in \mathbb{Z}_+^k$ length of the corresponding interval $\gamma(J_k)$ is bounded by $l(\gamma(J_k)) \leqslant 100^{-k}$;

(b) for each $l > k$ we have $\mu_l(\gamma(J_k)) = \mu_k(\gamma(J_k))$;

(c) $\frac{d\mu_{k+1}}{d\mu_k}(x) \leqslant 1 + \delta$ for every point $x \in \bigcup_{J_k \in \mathbb{Z}_+^k} \gamma(J_k)$.

Let $\mu = \lim_{k\to\infty} \mu_k$ in the sense of weak limit. Then $\mathrm{HD}(\mu) \geqslant d(\delta)$, where $d(\delta) \to 1$ as $\delta \to 0$.

**Lemma 8.** For each $\delta > 0$ there exists $\theta > 0$ such that the densities of $\frac{d\mu_{k+1}}{d\mu_k}(x)$ used to define measures $\mu_{k+1}$ knowing $\mu_k$ satisfy condition (c) of Lemma 7.

*Proof of Lemma 7.* We prove that for any segment $I$ we have $\mu(I) \leqslant \mathrm{Const}\, |I|^{1-\beta}$, where $\beta \to 0$ as $\delta \to 0$. Let $k(I) = \frac{|\ln |I||}{\ln 100}$ and $a$ and $b$ be the endpoints of $I$. Let $\tilde{a}$ be the left endpoint of the $k$-th partition containing $a$ and $\tilde{b}$ be the right endpoint of the $k$-th partition containing $b$. Then

$$\mu(I) \leqslant \mu([\tilde{a}, \tilde{b}]) = \mu_k([\tilde{a}, \tilde{b}]) \leqslant (1+\delta)^k \, \mu_0([\tilde{a}, \tilde{b}]) \leqslant 3(1+\delta)^k \, |I| \leqslant 3\,|I|^{1-\beta} \tag{35}$$

where $\beta = \frac{\ln(1+\delta)}{\ln 100}$. Thus, $\beta \to 0$ as $\delta \to 0$. Application of the mass distribution principle implies that $\mathrm{HD}(\mu) \geqslant 1-\beta$. ▌

*Proof of Lemma 8.* Recall the notation of Section 6. We need to show that

$$\sup_{J_{k+1}} \frac{|\varphi^{-1}_{n^{(k+1)}} \sigma(J_{k+1})|}{|\varphi^{-1}_{n^{(k)}} \gamma(J_{k+1})|} \to 1, \qquad \theta \to 0 \tag{36}$$

By construction

$$\frac{|\varphi_{n^{(k+1)},\, n^{(k)}} \sigma(J_{k+1})|}{|\gamma(J_{k+1})|} \geqslant 1 - \left(\frac{\theta}{r/100}\right).$$

Hence to prove (36) it is enough to show that there is a constant $C$ independent of $j$, $k$, $l$ such that for any interval $I \subset \gamma(J_{k+1})$

$$\frac{|\varphi^{-1}_{n_j^{(k)}} I|}{|\varphi^{-1}_{n_j^{(k)}} \gamma(J_{k+1})|} \leqslant C \, \frac{|I|}{|\gamma(J_{k+1})|}.$$

To do so it is enough to show that there is a constant $\bar{C}$ such that for every pair $y_1, y_2 \in \gamma(J_{k+1})$

$$\frac{\| d\varphi^{-1}_{n_j^{(k)}} \mid T\gamma(J_{k+1}) \| \, (y_1)}{\| d\varphi^{-1}_{n_j^{(k)}} \mid T\gamma(J_{k+1}) \| \, (y_2)} \leqslant \bar{C}.$$

But by Proposition 6 there are constants $C_1$, $C_2$, and $C_3$ such that

$$\begin{aligned}
&|\ln \| d\varphi^{-1}_{n_j^{(k)}} \mid T\gamma(J_{k+1}) \| \, (y_1) - \ln \| d\varphi^{-1}_{n_j^{(k)}} \mid T\gamma(J_{k+1}) \| \, (y_2)| \\
&\quad \leqslant \sum_{m=1}^{k} |\ln \| d\varphi_{n^{(m)},\, n^{(m-1)}} \mid T\varphi_{n^{(m)}} \| \, (\varphi_{n^{(k)},\, n^{(m)}}\, y_1) \\
&\qquad - \ln \| d\varphi_{n^{(m)},\, n^{(m-1)}} \mid T\varphi_{n^{(m)}} \| \, (\varphi_{n^{(k)},\, n^{(m)}}(y_2))| \\
&\quad \leqslant C_1 \sum_m \rho^{\alpha}(\varphi_{n^{(k)},\, n^{(m)}}\, y_1, \varphi_{n^{(k)},\, n^{(m)}}(y_2)) \leqslant C_2 \sum_m 100^{(m-k)\alpha} \rho^{\alpha}(y_1, y_2) \leqslant C_3 r^{\alpha}.
\end{aligned}$$

This completes the proof. ▌

## 8. PROOF OF THEOREM 4

Let $\mathcal{G}$ denote the foliation of $\mathbb{T}^N$ by curves

$$\{x^1 = c^1, x^2 = c^2 \dots x^{N-1} = c^{N-1}\}.$$

By (35) for each $\beta > 0$ and each leaf $\gamma_c$ of $\mathcal{G}$ almost surely there exists a measure $\mu_c$ on $\gamma_c$ such that $\mu_c(I) \leqslant 3\,|I|^{1-\beta}$ and $\mu_c(\mathbf{L}_\theta) = 1$. Let $\mu = \int \mu_c\, dc$. Then by Fubini Theorem almost surely for any cube $\mathcal{C}$ of side $r$ we have $\mu(\mathcal{C}) \leqslant 3r^{N-\beta}$ and $\mu(\mathbf{L}_\theta) = 1$. The application of the mass distribution principle completes the proof. ▌

## APPENDIX A. LARGE DEVIATIONS

Here we collect some estimates used throughout the proof of Theorem 3.

**Lemma 9.** Let $\mathcal{F}_j$ be a filtration of $\sigma$-algebras and $\{\xi_j,\}$ be a sequence of $\mathcal{F}_j$-measurable random variables such that

(a) there exist $C_1, \lambda$ such that for every $|s| \leqslant \lambda$ we have $\mathbb{E}(e^{s\xi_{j+1}} \mid \mathcal{F}_j) \leqslant C_1$;

(b) there exists $C_2$ such that $\mathbb{E}(\xi_{j+1} \mid \mathcal{F}_j) \leqslant C_2$.

Then for each $\epsilon > 0$ the probability

$$\mathbb{P}\left\{\sum_{j=0}^{N-1} \xi_j \geqslant (C_2 + \epsilon)\, N\right\}$$

decays exponentially in $N$.

*Proof.* Consider

$$\Phi_n(s) = \exp\left\{\left(\sum_{j=0}^{n-1} \xi_j - \left(C_2 + \frac{\epsilon}{2}\right) n\right) s\right\}.$$

Then (a) and (b) imply that $\Phi_n(s)$ is a supermartingale if $s$ is sufficiently small. Hence $\mathbb{E}\Phi_n(s) \leqslant \mathbb{E}\Phi_0(s) = 1$, and so

$$\mathbb{E} \exp\left\{\left(\sum_{j=0}^{n-1} \xi_j - (C_2 + \epsilon)\, n\right) s\right\} \leqslant \exp\left(-\frac{n\epsilon s}{2}\right),$$

which proves the lemma. ▌

**Lemma 10.** Let $\mathscr{F}_j$ be a filtration of $\sigma$-algebras and $\{\xi_j,\}$ be a sequence of $\mathscr{F}_j$-measurable random variables such that there exists constant $C$ such that

$$\mathbb{E}(\xi_{j+1} \mid \mathscr{F}_j) \leqslant C \tag{37}$$

then for every $\epsilon, \varepsilon > 0$ there is $R > 0$ such that

$$\mathbb{P}\left\{\frac{\#\{n \leqslant N : \xi_j \leqslant Re^{\epsilon(n-j)} \text{ for all } 0 \leqslant j < n\}}{N} \leqslant 1-\varepsilon\right\}$$

tends to zero exponentially fast in $N$.

*Proof.* We say that a pair $(j, n)$ is $R$-bad if

$$\xi_j > Re^{\epsilon(n-j)}.$$

By (37)

$$\mathbb{P}\{(j, n) \text{ is } R\text{-bad}\} \leqslant \frac{C}{R} e^{-\epsilon(n-j)}. \tag{38}$$

Now given $k$ let $B_R(k)$ be the number of $n > k$ such that $(k, n)$ is $R$-bad. By (38)

$$\mathbb{E}(B_R(k+1) \mid \mathscr{F}_k) \leqslant \frac{Ce^{-\epsilon}}{R(1-e^{-\epsilon})} \to 0$$

as $R \to \infty$. Thus by Lemma 9 there exists $R$ such that

$$\mathbb{P}\left\{\sum_{k=1}^{N} B_R(k) \geqslant \varepsilon N\right\}$$

decays exponentially in $N$. This completes the proof of the lemma. ∎

**Proposition 8.** Let $x_j$ be (in general, non-homogeneous) random walk on $\mathbb{Z}$. Suppose that there exist constants $C_1, C_2, C_3$ such that

(a) there exist $m$ such that for every $x_j > m$ we have $\mathbb{E}(x_{j+1} - x_j \mid x_j) \leqslant -C_1$;

(b) for every $x_j$ and every $\zeta < C_2$ we have $\mathbb{E}(e^{\zeta(x_{j+1}-x_j)} \mid x_j) \leqslant C_3$.

Fix $\delta > 0$. Let $F(M)$ denote the set of $j$ such that for all $k < j$

$$x_k \leqslant \max(x_j, m) + M + \delta(j-k).$$

Then for every $\varepsilon > 0$ there exists $M > 0$ such that

$$\mathbb{P}\left\{\frac{\#\{F(M)\cap[1,N]\}}{N} \leqslant 1-\varepsilon\right\}$$

decays exponentially in $N$.

*Proof.* Let $\tau_1 < \tau_2 < \cdots < \tau_k < \cdots$ be the consecutive returns of $x_j$ to $\{x \leqslant m\}$. Let

$$t_j = \tau_{j+1} - \tau_j, \qquad X_j = \max_{\tau_{j-1} < l < \tau_j} x_l.$$

**Lemma 11.** $t_j$ and $X_j$ have exponential tails.

*Proof.* It suffices to prove it for $t_1$ and $X_1$ and the assumption that $x_0 \leqslant m$. Clearly it suffices to condition on $x_1 > m$ since otherwise $t_1 = 1$, $X_1 \leqslant m$. Then (b) implies that for small $\varepsilon_1$, $\varepsilon_2$

$$y_j = e^{\varepsilon_1 x_j + \varepsilon_2 j} 1_{\{j \leqslant \tau_1\}}$$

is a supermartingale. Thus

$$\mathbb{E} y_j \leqslant \mathbb{E} y_1 \leqslant C_4. \tag{39}$$

On the other hand

$$\mathbb{E} y_j \geqslant \mathbb{P}\{\tau_1 > j\}\, e^{\varepsilon_1 m + \varepsilon_2 j}.$$

Hence

$$\mathbb{P}\{\tau_1 > j\} \leqslant C_5 e^{-\varepsilon_2 j}$$

where $C_5 = C_4 e^{-\varepsilon_1 m}$. Now

$$\mathbb{E} e^{\varepsilon_1 X_1} \leqslant \mathbb{E}\left(\sum_{j=1}^{\tau_1} e^{\varepsilon_1 x_j}\right) \leqslant \sum_{j=1}^{\infty} \mathbb{E} e^{\varepsilon_1 x_j} \sum_{k=j}^{\infty} \mathbb{P}\{\tau_1 > k\} \leqslant C_4 \sum_j \frac{C_5 e^{-\varepsilon_2 j}}{1 - e^{-\varepsilon_2 j}} < \infty.$$

This completes the proof. ▮

The rest of the proof of Proposition 3 is similar to the proof of Lemma 10. We say that the pair $(k, j)$ is bad if

$$x_k > \max(x_j, m) + M + \delta(j-k).$$

If $(k, j)$ is bad then

$$j-k<\frac{x_k-m+M}{\delta}.$$

Let $B_l(M)$ be the number of bad pairs $(k, j)$ such that $\tau_{l-1}<k<\tau_l$. By the previous lemma $\mathbb{E}B_l(M)<\infty$ and so by dominated convergence theorem $\mathbb{E}B_l(M)\to 0$ as $M\to\infty$. Hence by Lemma 9 the number of bad pairs such that $k<\tau_N$ is less than $\varepsilon N$ except on a set of exponentially small probability. Since $\tau_N\geqslant N$ the proposition follows. ▮

## ACKNOWLEDGMENTS

D.D. was partially supported by NSF and Sloan Foundation, V.K. was partially supported by American Institute of Mathematics Fellowship and Courant Institute, and L.K. was partially supported by NSF postdoctoral fellowship.

## REFERENCES

1. D. Dolgopyat, V. Kaloshin, and L. Koralov, *Sample path properties of stochastic flows*, preprint.
2. M. Cranston, M. Schuetzow, and D. Steinsaltz, Linear expansion of isotropic Brownian flows, *El. Comm. Prob.* **4**:91–101 (1999).
3. R. Carmona, Transport properties of Gaussian velocity fields, in *Real and Stochastic Analysis*, M. M. Rao, ed., Probab. Stochastics Ser. (CRC, Boca Raton, FL, 1997), pp. 9–63.
4. R. A. Carmona and F. Cerou, Transport by incompressible random velocity fields: simulations & mathematical conjectures, in *Stochastic Partial Differential Equations: Six Perspectives*, R. Carmona and B. Rozovskii, ed., Math. Surveys Monogr., Vol. 64, (Amer. Math. Soc., Providence, RI, 1999), pp. 153–181.
5. R. Carmona, S. Grishin, L. Xu, and S. Molchanov, Surface stretching for Ornstein Uhlenbeck velocity fields, *El. Comm. Prob.* **2**:1–11 (1997).
6. M. Cranston and M. Schuetzow, *Dispersion rates for Kolmogorov flows,* preprint.
7. M. Cranston, M. Schuetzow, and D. Steinsaltz, Linear bounds for stochastic dispersion, *Ann. Prob.* **28**:1852–1869 (2000).
8. D. Dolgopyat, V. Kaloshin, and L. Koralov, *A limit shape theorem for periodic stochastic dispersion,* preprint.
9. H. Lisei and M. Scheutzow, Linear bounds and Gaussian tails in a stochastic dispersion model, *Stoch. Dyn.* **1**(3):389–403 (2001).
10. M. Schuetzow and D. Steinsaltz, *Chasing balls through martingale fields,* preprint.
11. C. Zirbel and E. Cinlar, Mass transport by Brownian flows, in *Stochastic Models in Geosystems*, S. A. Molchanov and W. A. Woyczynski, ed., IMA Vol. Math. Appl., Minneapolis, MN, 1994, Vol. 85, (Springer, New York, 1997), pp. 459–492.
12. C. Zirbel and E. Cinlar, Dispersion of particle systems in Brownian flows, *Adv. in Appl. Prob.* **28**:53–74 (1996).

13. P. Baxendale and D. Stroock, Large deviations and stochastic flows of diffeomorphisms, *Prob. Th. & Rel. Fields* **80**:169–215 (1988).
14. A. Carverhill, Flows of stochastic dynamical systems: ergodic theory, *Stochastics* **14**:273–317 (1985).
15. P. Baxendale, Lyapunov exponents and relative entropy for a stochastic flow of diffeomorphisms, *Probab. Th. & Rel. Fields* **81**:521–554 (1989).
16. Ya. Pesin, Characteristic Liapunov exponents, and smooth ergodic theory, *Russ. Math. Surveys* **32**:55–114 (1977).
17. A. Katok, Lyapunov exponents, entropy and periodic orbits for diffeomorphisms, *Publ. IHES* **51**:137–173 (1980).
18. S. Newhouse, Continuity properties of entropy, *Ann. of Math.* **129**:215–235 (1989).
19. D. Dolgopyat, Bounded orbits of Anosov flows, *Duke Math. J.* **87**:87–114 (1998).
20. D. Kleinbock and G. Margulis, Bounded orbits of nonquasiunipotent flows on homogeneous spaces, in *Sinaĭ's Moscow Seminar on Dynamical Systems*, L. A. Bunimovich, B. M. Gurevich, and Ya. B. Pesin, ed., Amer. Math. Soc. Transl. Ser. 2, Vol. 171 (Amer. Math. Soc., Providence, RI, 1996), pp. 141–172.
21. L. Barreira and J. Schmeling, Sets of "non-typical" points have full topological entropy and full Hausdorff dimension, *Israel J. Math.* **116**:29–70 (2000).
22. D. Szasz, Ball-avoiding theorems, *Erg. Th. & Dyn. Sys.* **20**:1821–1849 (2000).
23. Ya. Pesin, *Dimension Theory in Dynamical Systems*, Chicago Lect. Math. (U. Chicago Press, Chicago/London, 1997).
24. J. Alves, SRB measures for non-hyperbolic systems with multidimensional expansion, *Ann. Sci. Ecole Norm. Sup.* **33**:1–32 (2000).
25. Ya. Pesin, Families of invariant manifolds that correspond to nonzero characteristic exponents, *Math. USSR-Izvestiya* **40**:1261–1305 (1976).
26. M. Jakobson, Absolutely continuous invariant measures for one-parameter families of one-dimensional maps, *Comm. Math. Phys.* **81**:39–88 (1981).
27. L. S. Young, Statistical properties of dynamical systems with some hyperbolicity, *Ann. of Math.* **147**:585–650 (1998).
28. J. Alves, C. Bonatti, and M. Viana, SRB measures for partially hyperbolic systems whose central direction is mostly expanding, *Invent. Math.* **140**:351–398 (2000).
29. D. Dolgopyat, On dynamics of mostly contracting systems, *Comm. Math. Phys.* **213**(1):181–201 (2000).
30. M. Hirsch, C. Pugh, and M. Shub, *Invariant Manifolds*, Lecture Notes in Mathematics, Vol. 583 (Springer, Berlin/New York, 1977).
31. M. Cranston and Y. Le Jan, Asymptotic curvature for stochastic dynamical systems, in *Stochastic Dynamics*, Bremen, 1997, H. Crauel and M. Gundlach, ed. (Springer, New York, 1999), pp. 327–338.
32. S. Lemaire, Invariant jets of a smooth dynamical system, *Bull. Soc. Math. France* **129**:379–448 (2001).
33. C. Pugh, M. Shub, and A. Wilkinson, Holder foliations, *Duke Math. J.* **86**:517–546 (1997); Correction: **105**:105–106 (2000).
34. M. Jiang, Ya. Pesin, and R. de la Llave, On the integrability of intermediate distributions for Anosov diffeomorphisms, *Erg. Th. & Dyn. Sys.* **15**:317–331 (1995).